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Coordinates Used in Derivation of Hawking Radiation via Hamilton-Jacobi Method

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Abstract Coordinates used in derivation of Hawking radiation via Hamilton-Jacobi method are investigated more deeply. In the case of a 4-dimensional Schwarzschild black hole, a direct computation leads to a wrong result. In the meantime, making use of the isotropic coordinate or invariant radial distance, we can get the correct conclusion. More coordinates including Painleve and Eddington-Finkelstein are tried to calculate the semi-classical Hawking emission rate. The reason of the discrepancy between naive coordinate and well-behaved coordinates is also discussed.

Keywords Black hole · Hawking radiation · Tunneling effect · Painleve coordinate · Eddington-Finkelstein coordinate

1 Introduction

In 1970s, it was discovered by Hawking [1, 2] that a classical black hole could radiate thermal spectrum of particles using the basic principles of quantum field theory. This discovery connects a black hole with thermodynamics, and a black hole's entropy depends on the surface area of its horizon as $S_{BH} = \frac{A}{4}$ [3–5]. However, Hawking radiation takes no information out from the black hole [6, 7] due to its black body spectrum, so it gives rise to a disturbing problem about information conservation.

After 1995, a description of black hole radiation and back reaction as a tunneling effect has been investigated semiclassically [8–14]. The essence for such calculations is the computation of radial trajectories [12, 15] in the static or stationary region representing the domain of outer communication of the black hole, except that an infinitesimal region behind the event horizon is allowed, which in fact plays a major role. It is impossible to travel from inside to infinity along classically permitted trajectories. In the tunneling approach instead, the particles are allowed to follow classically forbidden trajectories, by starting just

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behind the horizon onward to infinity. Their result is considered possibly in agreement with information conservation. In their method, energy conservation and WKB approximation $\Gamma = e^{-2\mathrm{Im}S}$ are used. Moreover, a corrected spectrum, which is approximated to the first grade, is given. Considering self-interaction, as a particle emits, the black hole loses energy. This means that the black hole will be "dynamical" and shrink as particles radiate, and this also could be called the emission particle's back-reaction. Following this method, many works [16–20] have been done about static and stationary black holes.

The computation of radial trajectories needs a well-behaved coordinate system, Painleve coordinate [21] was almost used in all the research papers. The Painleve coordinate system has many attractive features. Firstly, the components of the metric in Painleve coordinate are regular, not diverging at the event horizon. Secondly, constant-time slices are just flat Euclidean space. The second one is very important because WKB approximation is applied to calculate the tunneling rate, and WKB approximation is only right in flat space. Thirdly, particle tunneling a barrier is an instantaneous process, Zhang and Zhao [22] suggested that it should satisfy Landau coordinate clock synchronization condition [23]. After some investigation, Ren and Zhao [24, 25] proposed that the main aim and the crucial point to introduce a new coordinate system is to eliminate the singularity of components of the metric at the event horizon. They applied well-known Eddington-Finkelstein coordinate instead of Painleve one, and have got the same correct results. They claimed that WKB approximation could be extended to the space which is not flat Euclidean. As a condition instead, the event horizon and time-like limit surface should be identical.

In recent two years, Liu et al. proposed that the same information conservation result can also be got using Damour-Ruffini method [26–29]. This method is not only simpler, but also has an explicit physical picture. Considering of the radiating particles' back reaction to black hole, when energy conservation, charge conservation, and momentum conservation are taken into account, the authors have obtained a possible explanation to information loss paradox.

In 2005, Angheben et al. [30] applied Hamilton-Jacobi equation and WKB approximation to deal with tunneling for extremal and rotating black holes without considering the particle's back-reaction effect. After some calculations, they found that the contribution due to the integral over the radial coordinate is divergent as soon as the integration includes horizon. One needs a regularization, the natural one is equivalent of the Feynman prescription and it consists in deforming the contour and, as it is well known, this produces an imaginary part, whose physical consequence is associated with tunneling process. However, thinking of a Schwarzschild black hole, the natural naive approach leads to an imaginary contribution which is one half the correct one. In the meantime, if one repeats the same calculation making use of the isotropic coordinate or invariant radial distance, a direct computation gives the correct result.

Because there is the similar coordinate problem in Hamilton-Jacobi method to calculate Hawking radiation, we want to discuss some more coordinate systems in this paper. The basic idea is possibly that the well-behaved coordinate systems in radial trajectory method can be used in Hamilton-Jacobi method. So, in Sect. 2, we will simply review the calculations using natural naive coordinate, isotropic coordinate and invariant radial distance. In Sect. 3, we show that Painleve coordinate can be used to get the same result. In Sect. 4, Eddington-Finkelstein coordinate is tried to calculate the action too. In the end, we will discuss the relation between the calculation in Hamilton-Jacobi method and different coordinate systems.

2 Hamilton-Jacobi Method and Tunneling at Event Horizon

The line element of Schwarzschild black hole is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

in which $f(r) = 1 - \frac{2M}{r}$, and its horizon is $r_h = 2M$.

We firstly consider the problem of a scalar particle moving in this space-time without its self-gravitation (or back-reaction). Within the semi-classical approximation, the classical action I of the particle satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\upsilon}\partial_\mu I \partial_\upsilon I + m^2 = 0, \tag{2}$$

where m is the mass of the scalar particle.

Considering of (1), (2) can be rewritten as

$$-\frac{1}{f(r)}(\partial_t I)^2 + f(r)(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi I)^2 + m^2 = 0. \tag{3}$$

As usual, due to the symmetries of the metric, we can suppose a solution as following form

$$I = -Et + W(r) + J(x^i),$$

therefore we have

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_\theta I = J_\theta, \qquad \partial_\varphi I = J_\varphi, \tag{4}$$

where J_θ and J_φ are constants respectively.

Then, putting (4) into (3), we can get the classical action of an outgoing particle

$$I = -Et + \int \frac{\sqrt{E^2 - f(r)(\frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\varphi^2 + m^2)}dr}{f(r)} + J(x^i). \tag{5}$$

If we directly use Feynman prescription to deal with the integral over the coordinate, we will get one half of the correct one, that is $\mathrm{Im}I = \mathrm{Im}W = \pi r_h E$.

However, if one repeats the above calculation making use of the isotropic coordinate defined by

$$t \to t,\ r \to \rho, \quad \ln\rho = \int \frac{dr}{r\sqrt{f(r)}}, \tag{6}$$

the metric assumes the form

$$\begin{aligned} ds^2 &= -f\big(r(\rho)\big)dt^2 + \frac{r^2(\rho)}{\rho^2}\big(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2)\big) \\ &= -dt^2\frac{(1-\frac{r_h}{4\rho})^2}{(1+\frac{r_h}{4\rho})^2} + \left(1 + \frac{r_h}{4\rho}\right)^4(d\rho^2 + \rho^2 dS_2^2). \end{aligned} \tag{7}$$

In this system of coordinate, the spatial metric is no longer singular at the horizon. This form of metric is still static, but with a radial part regular at the horizon $\rho = r_h$. We may apply again similar formula as (5) deforming the contour and a direct computation gives the correct result $\mathrm{Im}I = \mathrm{Im}W = 2\pi r_h E$.

The reason of this discrepancy can be understood observing that in a curved manifold, the non-locally integrable function $1/r$ does not lead to a covariant distribution $1/(r + i0)$. Because the result above is not invariant under changes of coordinate within a time slice, we introduce the proper spatial distance defined by the spatial metric

$$d\sigma^2 = f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{8}$$

so the radial part of the action can read

$$W(\sigma) = \int \frac{d\sigma\sqrt{E^2 - f(r(\sigma))(\frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\varphi^2 + m^2)}}{\sqrt{f(r(\sigma))}}. \tag{9}$$

Using the following near-horizon approximation

$$f(r) = f'(r_h)(r - r_h) + \cdots,$$

we get the invariant result

$$I = \frac{2\pi i E}{f'(r_h)} + (\text{real contribution}) = 4\pi i M E + (\text{real contribution}), \tag{10}$$

and the semi-classical emission rate reads

$$\Gamma = e^{-2\mathrm{Im}I} = e^{-8\pi M E}. \tag{11}$$

So we can easily get the Boltzmann factor

$$\beta = 8\pi M.$$

3 Painleve Coordinate

To describe tunneling from the black hole, we must use a coordinate system that is regular at the horizon. Painleve line element is as follows

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + 2\sqrt{\frac{2M}{r}}dtdr + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{12}$$

Using the coordinate (12), (2) can be written as

$$-\partial_t I \partial_t I + 2\sqrt{\frac{2M}{r}}\partial_t I \partial_r I + \left(1 - \frac{2M}{r}\right)(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi I)^2 + m^2 = 0. \tag{13}$$

Considering symmetries of the metric, we can easily obtain a solution as the following

$$I = -Et + W(r) + J(x^i),$$

in which E is the energy of the particle.

Therefore, we can get

$$\partial_t I = -E, \qquad \partial_r I = W'(r), \qquad \partial_i I = J_i, \tag{14}$$

where J_i are constants.

Putting (14) into (13), we can get

$$\left(1 - \frac{2M}{r}\right)\left(W'(r)\right)^2 - 2E\sqrt{\frac{2M}{r}}W'(r) + \alpha = 0, \tag{15}$$

where

$$\alpha = \frac{1}{r^2}J_\theta^2 + \frac{1}{r^2\sin^2\theta}J_\varphi^2 + m^2 - E^2,$$

and we can find that α is a constant at the horizon.

The action of an outgoing particle can be written as

$$I = -Et + \int \frac{Er(1 + \sqrt{\frac{2M}{r} - \frac{\alpha}{E}(1 - \frac{2M}{r})})dr}{r - 2M} + J(x^i). \tag{16}$$

We use Feynman prescription to calculate the integral, and get the result

$$I = -Et + i4\pi ME + J(x^i). \tag{17}$$

Now we can get the tunneling rate using WKB approximation

$$\Gamma \sim e^{-2\mathrm{Im}I} = e^{-8\pi ME} = e^{-\beta E}, \tag{18}$$

and

$$\beta = \frac{1}{T} = 8\pi M,$$

where $T = \frac{1}{8\pi M}$ is Hawking temperature of the black hole.

4 Eddington-Finkelstein Coordinate

To describe tunneling at the horizon, we can also try to use Eddington-Finkelstein coordinate, which is also regular at the event horizon. The line element is as the following

$$ds^2 = -\left(1 - \frac{2m}{r}\right)d\nu^2 + 2d\nu dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{19}$$

in which $\nu = t + r_*$ and $r_* = r + 2M\ln(\frac{r-2M}{2M})$.

Due to symmetries of the metric, the action can similarly be written as

$$I = -E\nu + \tilde{W}(r) + \tilde{J}(x^i). \tag{20}$$

Using Hamilton-Jacobi method as above, the action corresponding to an outgoing particle is given as

$$I = -E\nu + \int \frac{Er(1 + \sqrt{1 - \frac{\lambda}{E}(1 - \frac{2M}{r})})dr}{r - 2M} + \tilde{J}(x^i) = -E\nu + i4\pi ME + \tilde{J}(x^i), \tag{21}$$

where

$$\lambda = \frac{1}{r^2}\tilde{J}_\theta^2 + \frac{1}{r^2 \sin^2\theta}\tilde{J}_\varphi^2 + m^2,$$

and

$$\partial_v I = -E, \qquad \partial_r I = W'(r), \qquad \partial_i I = \tilde{J}_i,$$

in which $\tilde{J}_i$ is a constant.

Therefore, we can also easily get the tunneling rate and Hawking temperature as same as (18). We find that it is necessary to use the "well-behaved" coordinates to describe the tunneling process.

Then, we consider the self-interaction of the particles: when a particle with energy ω_i emits through the horizon, due to energy conservation, the mass of black hole M will be reduced to $(M - \omega_i)$, so the event horizon will change from $r_h^i = 2M$ to $r_h^f = 2(M - \omega_i)$. The emission rate should be rewritten as

$$\begin{aligned}\Gamma &= \prod_i \Gamma_i = \prod_i e^{-8\pi(M-\omega_i)\omega_i} = e^{\sum_i -8\pi(M-\omega_i)\omega_i} \\ &= e^{\int_0^E -8\pi(M-\omega')d\omega'} = e^{-4\pi(2M-E)E} = e^{-8\pi ME(1-\frac{E}{2M})} = e^{\Delta S_{BH}},\end{aligned} \tag{22}$$

where considering the semiclassical approximation, we use integral instead of the sum, and $\Delta S_{BH} = 4\pi[(M-E)^2 - M^2]$. So we will find that the spectrum of the tunneling is no longer purely thermal and this result is consistent with that of Parikh and Wilczek.

5 Discussions and Conclusions

Coordinates used in derivation of Hawking radiation via Hamilton-Jacobi method are investigated more deeply. In the case of a 4-dimensional Schwarzschild black hole, a direct computation leads to a wrong result. In the meantime, making use of the isotropic coordinate or invariant radial distance, we can get the correct conclusion. More coordinates including Painleve and Eddington-Finkelstein are tried to calculate the semi-classical Hawking emission rate. The reason of the discrepancy between natural naive coordinate and well-behaved coordinates can be understood observing that in a curved manifold, the non-locally integrable function $1/r$ does not lead to a covariant distribution $1/(r + i0)$. One has to make use of the invariant distance or change the metric into some well-behaved coordinates. After calculations, we have found that Painleve and Eddington-Finkelstein coordinates are all well-behaved. Maybe that the components of the metric in the coordinates should be all regular at the event horizon and there is a time-like Killing vector are the enough conditions to be applied in Hawking radiation calculation via Hamilton-Jacobi method. This is the same as the conclusions in [24, 25].

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